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Monotonic Games are Spanning Network Games

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Abstract: Spanning network games, which are a generalization of minimum cost spanning tree games, were introduced by Granot and Maschler (1991), who showed that these games are always monotonic. In this paper a subclass of spanning network games is introduced, namely simplex games, and it is shown that every monotonic game is a simplex game. Hence, the class of spanning network games coincides with the class of monotonic games.

1 Introduction

Cooperative game theory often models classes of 'interactive situations', where this term may have different interpretations. In such cases, it is important to identify the class of the mathematical models, namely the class of games that corresponds to the class of interactive situations. For example, von Neumann and Morgenstern (1944) modeled the class of 'interactive situations' that fall under the category of strategic form games in an environment of transferable utility and full and unrestricted cooperation as a class of TU-cooperative games. They proved (in both directions) that the corresponding games constitute precisely the class of superadditive TU-games. Borm and Tijs (1992) showed that the class of strategic form games in a non-transferable utility setting where players are allowed to coordinate their actions corresponds to the class of superadditive NTU-cooperative games that satisfy the requirement of standardness. As another example, the class of weighted majority constant sum games was identified in Peleg (1968) as a class of simple TU-games satisfying certain Bondareva-Shapley conditions. The class of exchange economies in a transferable utility environment, for example, was modeled and identified in Shapley and Shubik (1969) as the class of totally balanced games. Kalai and Zemel (1982) showed that the class of flows with private ownership corresponds to the class of non-negative totally balanced games. Curiel, Derks and Tijs (1989) showed that the class of flows, where the arcs are controlled by coalitions with veto players corresponds to the class of non-negative balanced games. Such identification is not always complete. For example, Curiel, Maschler and Tijs (1987) showed that bankruptcy situations give rise to non-negative convex games, but they also showed that not every non-negative convex game can be derived from a bankruptcy situation.

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One reason why such identification is important is, because it enables one to decide whether a certain solution concept is appropriate to the corresponding situation. For example, if the interactive situations correspond to the class of simple games, then it makes no sense to recommend the Shapley value on the basis of Shapley's (1953) original axioms, because the sum of simple games, which appears in one of the axioms, is not a simple game and therefore the system is not meaningful when restricted to that class. In fact, Dubey (1975) showed that another set of axioms, which does make sense in the restricted class of simple games, is sufficient to characterize the Shapley value. Now, if one wants to recommend and justify a solution for a class of interactive situations that corresponds to the class of simple games, one can refer to Dubey's axioms and check if they are appealing in the actual case under consideration.

The present paper is concerned with the identification of the class of TU-games that correspond to a class of *spanning network enterprises*. This class (see section 2) was defined by Granot and Maschler (1991) and it generalizes Megiddo's (1978) spanning tree and the more general Granot and Huberman (1981) monotonic minimal spanning network enterprises. Granot and Maschler (1991) prove that the games that result from these enterprises are monotonic. Here we shall, in reply to a question that was raised by Pradeep Dubey, prove that the converse is also true: for every monotonic TU-game (N, v), there is a spanning network enterprise whose game is (N, v).

In section 2 we formally introduce the model of a spanning network enterprise and its corresponding spanning network game. Further, some properties of spanning network games are investigated in this section. In section 3 a subclass of spanning network games, the class of so-called simplex games, is introduced and our main result is proved, namely that every monotonic game is a simplex game.

2 Spanning Network Games

A *spanning network enterprise* is a structure $\mathscr{S} := (V, E, a, b, N)$, where (V, E) is a finite undirected graph containing a distinguished vertex, 0, called the *root* or the *central supplier*. We assume that the graph (V, E) is connected. Further, a is a function from E to $\mathbb{R}$ that associates with each edge $e \in E$ a cost $a(e)$, and b is a function from V to $\mathbb{R}$ associating with each vertex $v \in V$ a cost $b(v)$. Note that both a and b can also assign negative values, in which case they represent profits rather than costs. In addition, $N = \{1, \dots, n\}$ is a set of *players*. Each player is located in a vertex $v \in V$. Vertices, other than the root, that are not inhabited by players will be called *switch boxes*. Note that we do not exclude the possibility that several players are located in the same vertex, neither did we exclude the possibility that the root is inhabited.

The players are users of some good that can be provided by the central supplier. Hence, the players in a *coalition* $S \subseteq N$ want to build a network that connects them to the root. Moreover, they want to do this in a cheapest possible way. Now, it is important to note that players are only located in vertices, they do not own them and cannot prevent other players from using the vertices they inhabit. Further, the players in S may find it profitable to build edges and vertices that they do not need for the actual connection to the root. They are allowed to do this. However, we do require that the network that is built by a coalition is connected and that it contains both end points of every edge it contains. Correspondingly, for every coalition $S \subseteq N$ the *cost* $c(S)$ is defined to be the cost of a least expensive connected subnetwork of (V, E) that connects all players in S to the root. Here, the cost of a subnetwork $G'=(V', E')$ is $w(G')=\sum_{v\in V'} b(v)+\sum_{e\in E'} a(e)$. Now, we defined a cost game $\Gamma_{\mathscr{S}}=(N, c)$ associated with a spanning network enterprise $\mathscr{S}$. The game $\Gamma_{\mathscr{S}}$ is called (the corresponding) *spanning network game*.

There is somewhat of a problem with the empty set of players. Should we require the empty coalition to pay the cost of the root? In this paper we will do so and we also allow the empty coalition to build profitable edges and vertices that are connected to the root. Note that this is consistent with the way we handle other coalitions. Further, it implies that for a spanning network game $\Gamma_{\mathscr{S}}=(N, c)$ it is not necessarily true that $c(\emptyset)$ equals zero. However, defining $c(\emptyset)$ to equal zero for every spanning network game would not change the results in this paper.

Example 1. Let $N=\{1, 2, 3\}$ and consider the network that is represented in figure 1, where the root is denoted by a triangle ∇ and switch boxes are denoted by a square $\square$, and where we omitted the costs that equal zero.

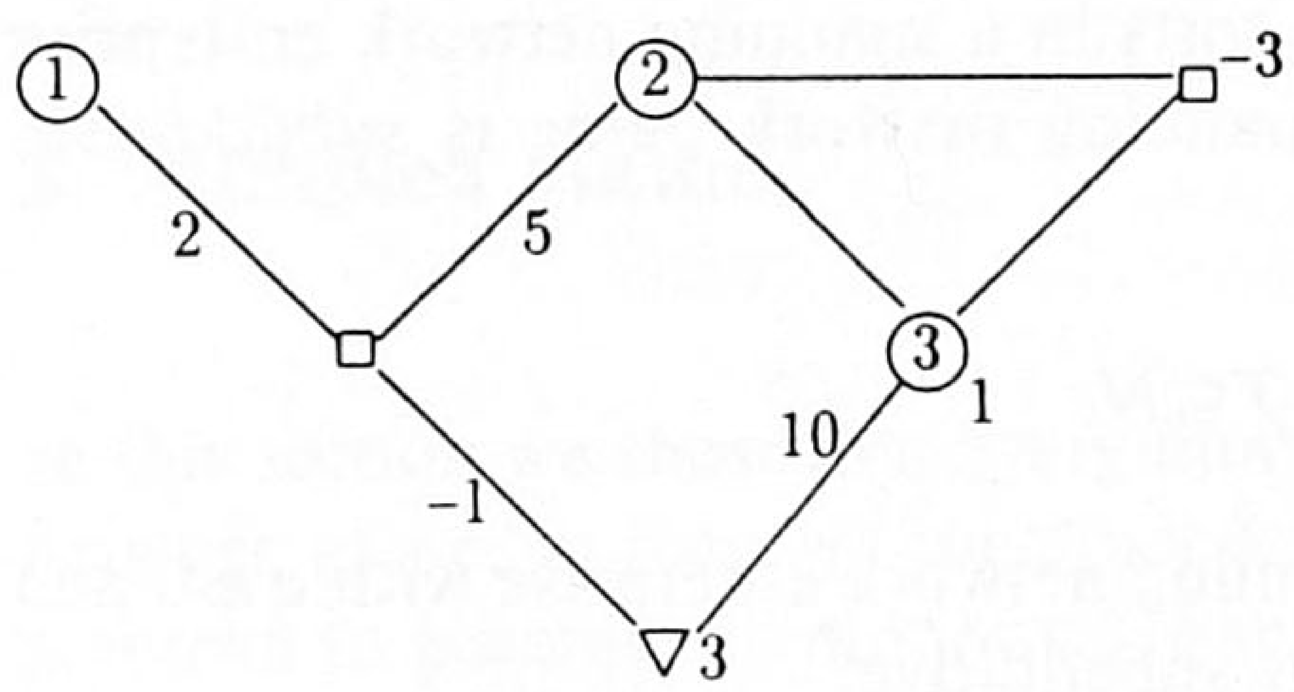

Fig. 1

Then the cost for the empty coalition is $c(\emptyset)=3-1=2$.

Further, an optimal subnetwork for the coalition $\{2\}$ is

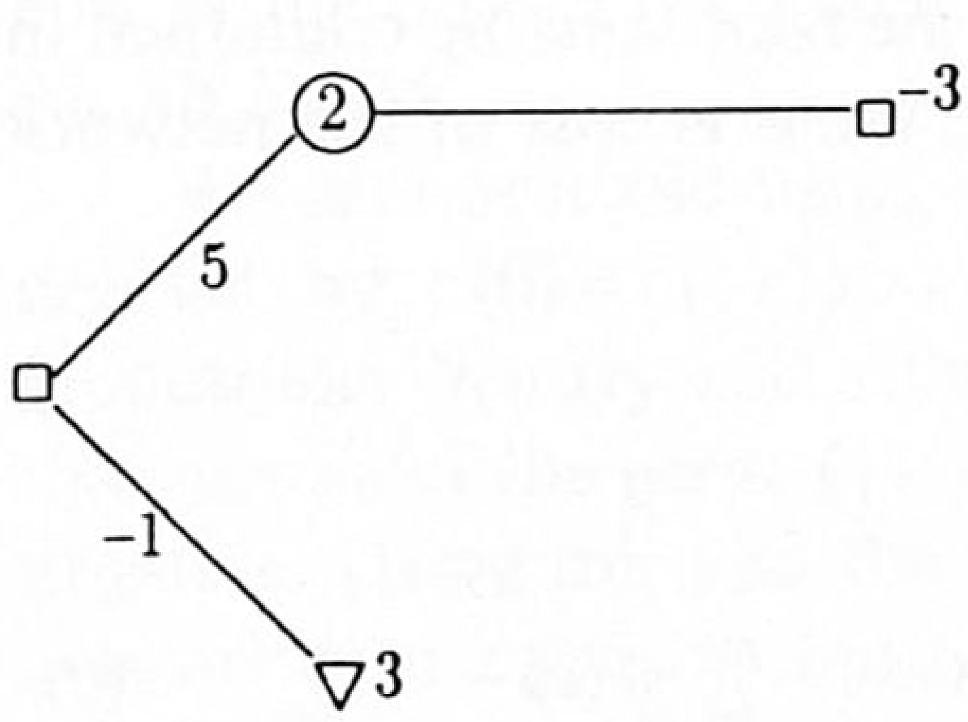

Fig. 2

Hence, for the spanning network game (N, c) associated with this network $c(\{2\})=3-1+5-3=4$. The game (N, c) is given in table 1.

Table 1

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	2	4	4	5	6	7	5	7

Now suppose in example 1 we lower the cost of the root by 2. Hence, the cost of the root is now 1 and all other costs are unchanged. Some calculation shows that the spanning network game (N, c_0) corresponding to this new situation is related to the game (N, c) in the following way: for each $S \subseteq N$ it holds that $c_0(S) = c(S) - 2$. So, in particular, $c_0(\emptyset) = 0$. This is a property that holds in general:

Remark 1. For every spanning network enterprise $\mathscr{S}$ and its associated game $\Gamma_{\mathscr{S}} = (N, c)$ we can define a slightly different enterprise $\mathscr{S}_0$ by lowering the cost of the root by $c(\emptyset)$ and the game $\Gamma_{\mathscr{S}_0} = (N, c_0)$ associated with this new enterprise satisfies $c_0(S) = c(S) - c(\emptyset)$ for all $S \subseteq N$, so especially, $c_0(\emptyset) = 0$ (this is a special instance of "Network Equivalence" as defined by Granot and Maschler (1991)).

Spanning network games are *monotonic*, i.e.

$$c(S) \leq c(T) \text{ for all } S \subseteq T \subseteq N,$$

as was shown by Granot and Maschler (1991).

The following theorem shows that if all costs in a spanning network enterprise are non-negative, then the corresponding spanning network game is *subadditive*, i.e.

$$c(S \cup T) \leq c(S) + c(T) \text{ for all disjoint } S, T \subseteq N.$$

Theorem 1. Let $\mathscr{S} = (V, E, a, b, N)$ be a spanning network enterprise with $a \geq 0$ and $b \geq 0$. Then the spanning network game $\Gamma_{\mathscr{S}}$ is subadditive.

Proof. Let $\Gamma_{\mathscr{S}} = (N, c)$ and $S, T \subseteq N$ such that $S \cap T = \emptyset$. Suppose $G^S = (V^S, E^S)$ is a subnetwork that is optimal for coalition S, i.e. G^S connects all players in S to the root and $w(G^S) = c(S)$, and let $G^T = (V^T, E^T)$ be an optimal subnetwork for coalition T. Consider the network $G^{S \cup T} := (V^S \cup V^T, E^S \cup E^T)$. This network obviously connects all players in $S \cup T$ to the root. Further, since the root must be contained in both V^S and V^T, the network $G^{S \cup T}$ is also connected. What is cost of the network $G^{S \cup T}$?

The inclusion-exclusion principle implies

$$\begin{aligned}
w(G^{S \cup T}) &= \sum_{v \in V^S \cup V^T} b(v) + \sum_{e \in E^S \cup E^T} a(e) \\
&= \sum_{v \in V^S} b(v) + \sum_{v \in V^T} b(v) - \sum_{v \in V^S \cap V^T} b(v) + \sum_{e \in E^S} a(e) + \sum_{e \in E^T} a(e) - \sum_{e \in E^S \cap E^T} a(e) \\
&= w(G^S) + w(G^T) - \sum_{v \in V^S \cap V^T} b(v) - \sum_{e \in E^S \cap E^T} a(e) \\
&\leq w(G^S) + w(G^T) = c(S) + c(T),
\end{aligned}$$

where the inequality follows from the fact that both $a \geq 0$ and $b \geq 0$.
This shows that the coalition $S \cup T$ has to spend no more than $c(S)+c(T)$ to build a connected subnetwork that connects the players of this coalition to the root. Hence, $c(S \cup T) \leq c(S)+c(T)$. □

The following example shows that spanning network games are not in general subadditive.

Example 2. Let $N=\{1,2\}$ and consider the network that is represented in figure 3.

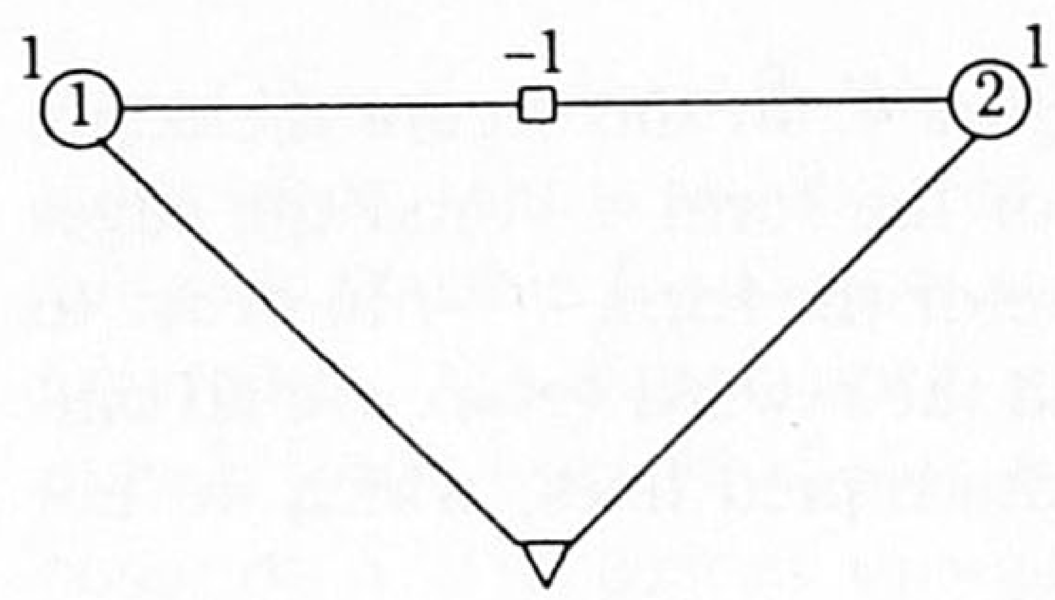

Fig. 3

The spanning network game (N, c) associated with this network satisfies $c(\{1\})=c(\{2\})=0$ and $c(N)=1$. This game is not subadditive.

3 Simplex Games

In this section we show that every monotonic transferable utility game is a spanning network game. In fact, we construct for each $n \in \mathbb{N}$ a so called simplex network, that is shown to generate all monotonic games with player set $\{1, \ldots, n\}$ just by adapting the costs of the vertices. Note that it suffices to consider monotonic games (N, c) with $c(\emptyset)=0$, because if we can find for each monotonic game (N, c) with $c(\emptyset)=0$, a spanning network enterprise $\mathscr{S}_{(N, c)}$ with corresponding spanning network game (N, c), then by remark 1 we can find for a monotonic game $(N, \tilde{c})$, with possibly $\tilde{c}(\emptyset) \neq 0$, a spanning network enterprise generating this game by adding $\tilde{c}(\emptyset)$ to the cost of the root in the spanning network enterprise $S_{(N, c^*)}$, where $c^*(S)=\tilde{c}(S)-\tilde{c}(\emptyset)$ for all $S \subseteq N$.

We cannot avoid using switch boxes. To see this, consider the game $(\{1, 2\}, c)$ defined by $c(\emptyset)=c(\{1\})=c(\{2\})=0$ and $c(\{1, 2\})=1$. Obviously, this game is monotonic. We try and find a spanning network enterprise without switch boxes that generates the game $(\{1, 2\}, c)$. Since $c(\emptyset)=0$, the cost of the root must be non-negative. Using this and the fact that $c(\{1\})=c(\{2\})=0$, we see that we can find a network that connects both 1 and 2 to the root and that has a non-positive cost. Hence, we cannot find a spanning network enterprise without switch boxes that generates the game $(\{1, 2\}, c)$.

In the following, let $n \in \mathbb{N}$ be fixed and $N := \{1, \ldots, n\}$. By e^S, $S \subseteq N$, we denote the vector in $\{0, 1\}^N$ that satisfies $e_i^S = 1 \Leftrightarrow i \in S$. The simplex network $\Delta_N = (V_N, E_N)$ is constructed as follows. The central supplier is identified with the origin, the vertex 0, and player i is identified with the vertex e^i, $i \in N$. Further, for each non-empty coalition $S \subseteq N$ there is a vertex d^S, the *door* for coalition S. For $\{i\} \subseteq N$ this door is the vertex $\frac{1}{2} e^i$, and for $S \subseteq N$ with $|S| \geq 2$ this door is the vertex $\dfrac{1}{|S|} e^S$, the center of gravity of the vertices e^i with $i \in S$. Finally, there is a *reward vertex R*, which is the vertex e^N. All edges in the simplex network are incident to a door d^S: for every non-empty $S \subseteq N$ door d^S is directly connected to the central supplier 0, to the reward vertex R, and to all e^i with $i \in S$.

The simplex network for $n = 3$ is sketched in figure 4. In this figure the edges connecting the doors d^S to the central supplier are of the form -·- and the edges connecting the doors d^S to the vertices e^i with $i \in S$ are of the form ---. In order to get a clear picture the edges between the doors d^S and the reward vertex are all omitted and, moreover, the unit cube is drawn in uninterrupted lines, which do not belong to the simplex network.

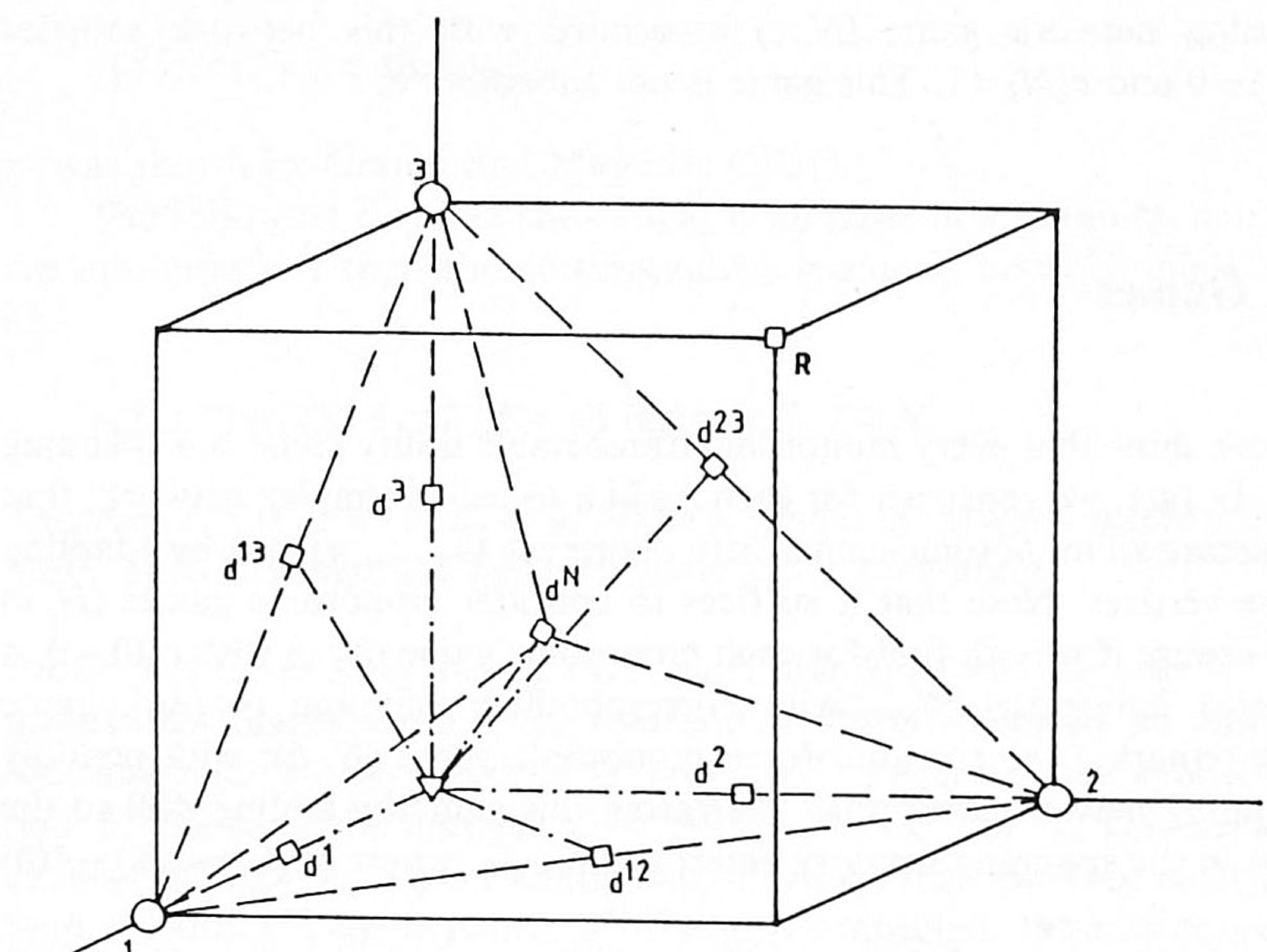

Fig. 4

We proceed by providing for every game (N, c) with $c(\emptyset) = 0$, a set of costs attached to the vertices of the simplex network Δ_N. Here we will use for a game (N, c) a *measure of non-subadditivity* of (N, c), $\alpha(c) \in \mathbb{R}^+$, which is defined by

$$a(c) := \max \left\{ 0, \max_{S \subseteq N: |S| \geq 2} \max_{(T_1, \ldots, T_k) \in P(S)} \left(\left(c(S) - \sum_{j=1}^{k} c(T_j) \right) (k-1)^{-1} \right) \right\},$$

where for $S \subseteq N$, $P(S)$ denotes the set of all partitions of S into at least two (non-empty) subcoalitions. Note that $\alpha(c)=0$ for subadditive games (N, c).

Now let (N, c) with $c(\emptyset)=0$, be fixed. We define all edges in the simplex network Δ_N to be costless, as well as the vertices corresponding to the central supplier and the single players. Further, for every coalition S, $S \neq \emptyset$, the cost of the vertex d^S, the door for coalition S, is $c(S)+\alpha(c)$, and to the reward vertex R we assign the cost $-\alpha(c)$ (the reward $\alpha(c)$). Now we have the following

Theorem 2. Let (N, c) be a monotonic game with $c(\emptyset)=0$. Then the simplex network Δ_N with costs as described above, generates the game (N, c).

Proof. Let $S \subseteq N$, $S \neq \emptyset$. We have to prove two things, namely that there is a subnetwork of Δ_N that is feasible for S and has cost $c(S)$, and that every subnetwork of Δ_N which is feasible for S costs at least $c(S)$.
Obviously, the subnetwork of Δ_N that is spanned by the central supplier, the door d^S, the vertices e^i with $i \in S$, and the reward vertex, is a feasible network for coalition S. The cost of this network is $c(S)+\alpha(c)+(-\alpha(c))=c(S)$.
Suppose $G=(V, E)$ is another subnetwork of Δ_N that is feasible for coalition S. Then this network has to contain the central supplier and the vertices e^i for each $i \in S$. Moreover, the network has to be connected. Since for each $i \in N$ the vertex e^i is only directly connected to the doors d^T with $i \in T \subseteq N$, the fact that G is connected implies that for every $i \in S$ there is a door d^{T_i} contained in G with $i \in T_i \subseteq N$. It is possible that we find the same door for different i. Hence, we can find $k \in \{1, \ldots, |S|\}$ and different $T_1, \ldots, T_k \subseteq N$ satisfying $S \subseteq \bigcup_{j=1}^{k} T_j$ such that G contains the doors $d^{T_1}, \ldots, d^{T_k}$. Using monotonicity of (N, c), it is clear that for all non-empty $T \subseteq N$ it holds that $c(T)+\alpha(c) \geq c(\emptyset)+\alpha(c)=\alpha(c) \geq 0$. Since the reward vertex is the only vertex having a non-positive cost $(-\alpha(c))$, we may without loss of generality assume that R belongs to G and hence, the cost $w(G)$ of the network G is at least

$$-\alpha(c)+\sum_{j=1}^{k}(c(T_j)+\alpha(c))=(k-1)\alpha(c)+\sum_{j=1}^{k}c(T_j). \tag{1}$$

We distinguish two cases.
If $k=1$, then $S \subseteq T_1$ must hold. Hence, by (1) and monotonicity of (N, c) it follows that

$$w(G) \geq c(T_1) \geq c(S).$$

If $k>1$, then we can find sets $T_1', T_2', \ldots, T_k'$ such that $T_j' \subseteq T_j$ for all $j \in \{1, \ldots, k\}$ and, moreover, $\{T_j' \mid j \in \{1, \ldots, k\}, T_j' \neq \emptyset\}$ forms a partition of S. One way to do this is define for all $j \in \{1, \ldots, k\}$

$$T_j' := (T_j \cap S) \setminus \bigcup_{i=1}^{j-1} T_i.$$

Now it follows from (1) and the monotonicity of (N, c) that

$$w(G)\geq \sum_{j=1}^{k} c(T_j)+(k-1)\,\alpha(c)\geq \sum_{j=1}^{k} c(T'_j)+(k-1)\,\alpha(c). \tag{2}$$

From the definition of $\alpha(c)$ we derive that

$$\alpha(c)\geq \left(c(S)-\sum_{j=1}^{k} c(T'_j)\right)(k-1)^{-1}. \tag{3}$$

Combining (2) and (3) shows that

$$w(G)\geq c(S).$$

This completes the proof of the theorem. □

A direct consequence of theorem 2 is

Corollary 3. The class of monotonic games coincides with the class of spanning network games.

Also, using the fact that $\alpha(c)=0$ for subadditive games (N, c) with $c(\emptyset)=0$, we derive from theorem 2:

Corollary 4. For every monotonic subadditive game (N, c) with $c(\emptyset)=0$ there is a spanning network enterprise $\mathscr{S}=(V, E, a, b, N)$ with $a\geq 0$ and $b\geq 0$ such that the associated spanning network game $\Gamma_{\mathscr{S}}$ equals (N, c).

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